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A Note on Convolution Theorem for Some Integral Transforms

Deepika Jain[1,a)], Alok Bhargava[1, b)], Kalpna Sharma[1, c)], Manoj Kumar[2, d)]

[1]Department of Mathematics & Statistics, Manipal University Jaipur, Jaipur, India
[2]Department of Computer and Communication Engineering, Manipal University Jaipur, Jaipur, India

[b)]*Corresponding author: alok.bhargava@jaipur.manipal.edu*
[a)]khandelwaldeepika5@gmail.com, [c)]kalpana.sharma@jaipur.manipal.edu,[d)]bohra.manoj1980@gmail.com

Abstract In the present paper, generalization of the convolution theorem for Sumudu Transform, Elzaki Transform, Mellin Transform and Natural Transform have been studied, respectively. Further, four corollaries have been derived showing that this study leads to the specific convolution property of the corresponding transformation by specifying parametric values.

Keywords: Convolution Theorem, Laplace Transform, Efros' theorem, Sumudu Transform, Elzaki Transform, Mellin Transform and Natural Transform

INTRODUCTION

The convolution $(f * g)(t)$ of two functions $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) = \int_0^t f(u)g(t-u)\ du$$

Graf [7] studied the generalization of the convolution theorem for Laplace Transform and stated the Efros' theorem [5] as under:
Let $L\{f(t)\} = F(s)$ and $L\{g(t,\rho)\} = G(s)e^{-q(s)\rho}$, then

$$F[q(s)]G(s) = L\left\{\int_0^\infty g(t,\rho)f(\rho)\ d\rho\right\} \tag{1}$$

where ρ is a parameter and

$$L\{f(t);s\} = F(s) = \int_0^\infty e^{-st} f(t)dt \quad ;Re(s) > 0$$

Further, on taking $q(s) = s$, (1) gives

$$G(s)e^{-s\rho} = L\{U(t-\rho)g(t-\rho)\} \tag{2}$$

which further leads to -

$$F[q(s)]G(s) = F(s)G(s) = L\left\{\int_0^\infty U(t-\rho)g(t-\rho)f(\rho)\ d\rho\right\}$$
$$= L\left\{\int_0^t g(t-\rho)f(\rho)\ d\rho\right\} = L\{(f * g)(t)\}$$
$$\Rightarrow F(s)G(s) = L\{(f * g)(t)\}$$

where, $U(t-\rho)$ is the unit step function defined as:

$$U(t-\rho) = \begin{cases} 1\ , & t > \rho \\ 0\ , & t < \rho \end{cases}$$

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This study, of course has provided a significant enhancement in the domain of study encompassing a wider range of problems of applied and computational mathematics which happens to be very important also towards filling the gap of previous work of this kind available in the literature.
Motivated by this work, here the authors have extended the concept to provide the generalization of convolution theorem for Elzaki Transform [3,9], Sumudu Transform [6,8], Natural Transform [2,12] and Mellin Transform [10,11].

Sumudu Transform

The Sumudu transform was introduced by Watugla [6] and some properties and applications were investigated by Belgacem and Karaballi [8]. The transform is derived from the classical Fourier integral and defined over the set of the functions

$$B = \{f(t)|\exists M, \rho_{1,}\rho_2 > 0 \text{ such that } |f(t)| < Me^{\frac{|t|}{\rho_j}}, t \in (-1)^j * [0,\infty)\} \tag{3}$$

The Sumudu transform of $f(t)$ is denoted by $S\{f(t)\} = F(u)$ and defined as

$$S\{f(t)\} = F(u) = \int_0^\infty f(ut)e^{-t}dt = \frac{1}{u}\int_0^\infty f(t)e^{\frac{-t}{u}}dt \quad ; u \in (\rho_1, \rho_2) \tag{4}$$

Elzaki Transform

The Elzaki transform[5] is defined over the set of functions

$B = \{f(t)|\exists M, \rho_{1,}\rho_2 > 0 \text{ such that } |f(t)| < Me^{\frac{|t|}{\rho_j}}, t \in (-1)^j * [0,\infty)\}$
The Elzaki transform of $f(t)$ is denoted by $E\{f(t)\} = F(u)$ and defined as

$$E\{f(t)\} = F(u) = u^2\int_0^\infty f(ut)e^{-t}dt = u\int_0^\infty f(t)e^{\frac{-t}{u}}dt \qquad ; u \in (\rho_1, \rho_2) \tag{5}$$

Mellin Transform

If $f: R_+ \to C$ is a function such that $f(t) = t^{u-1} \in L^1(R_+)$ for some $u \in C$, where $u = c + it$ and $c \in R, t \in R_+$ then the Mellin Transform [10] of the function $f(t)$ is defined by

$$M\{f(t)\} = F(u) = \int_0^\infty f(t)t^{u-1}dt \tag{6}$$

Natural Transform

The Natural transform [2] of the functions Related to a class B, where $B = \{f(t)|\exists M, \rho_{1,}\rho_2 > 0$ such that $|f(t)| < Me^{\frac{|t|}{\rho_j}}, t \in (-1)^j * [0,\infty)\}$. The Natural Transform of the function $f(t) > 0$ and $f(t) = 0$ for $t < 0$ is defined by

$$N^+\{f(t)\} = F(s,u) = \int_0^\infty e^{-st}f(ut)dt = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}}f(t)dt \quad ; s > 0, u > 0 \tag{7}$$

MAIN RESULTS

In this section of paper, the generalizations of convolution theorems of Sumudu Transform, Elzaki Transform, Mellin Transform and Natural transform has been developed.
Theorem 2.1: Let $f(t)$ and $g(t,\rho)$ are two functions from the set B, defined in (3) such that their Sumudu transforms are $S\{f(t)\} = F(u)$ and $S\{g(t,\rho)\} = G(u)e^{\frac{-\rho}{q(u)}}$, where ρ is a parameter, then

$$uF[q(u)]G(u) = S\left\{\int_0^\infty g(t,\,\rho)f(\rho)\ d\rho\right\} \tag{8}$$

Proof: In order to establish the theorem, from $f(t)$ and $g(t,\rho)$ forming a function

$$h(t) = \int_0^\infty g(t,\, \rho) f(\rho)\; d\rho \tag{9}$$

and taking Sumudu Transform

$$S\{h(t)\} = H(u) = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}}\, h(t)\, dt = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} \left\{\int_0^\infty g(t,\, \rho) f(\rho)\; d\rho\right\} dt$$

By changing the order of integrals

$$H(u) = \frac{1}{u}\int_0^\infty f(\rho) \left\{\int_0^\infty e^{\frac{-t}{u}}\, g(t,\, \rho) dt\right\} d\rho = \int_0^\infty f(\rho) S\{g(t,\, \rho)\}\, d\rho$$

$$= \int_0^\infty f(\rho) G(u)\, e^{\frac{-\rho}{q(u)}} d\rho = G(u)\int_0^\infty f(\rho)\, e^{\frac{-\rho}{q(u)}} d\rho = uF[q(u)]G(u)$$

Corollary 2.1: On taking $q(u) = u$ in (9)

$$G(u)e^{-\frac{\rho}{q(u)}} = G(u)e^{-\frac{\rho}{u}} = S\{U(t-\rho)g(t-\rho)\}$$

Which further leads to-

$$uF[q(u)]G(u) = uF(u)G(u) = S\left\{\int_0^\infty U(t-\rho)g(t-\rho)f(\rho)\, d\rho\right\}$$

$$= S\left\{\int_0^t g(t-\rho)f(\rho)\, d\rho\right\} = S\{(f*g)(t)\}$$

Which is the result of convolution theorem for the Sumudu Transform [6].

Theorem 2.2: Let $E\{f(t)\} = F(u)$ and $E\{g(t,\rho)\} = G(u)e^{\frac{-\rho}{q(u)}}$ are Elzaki transforms of the functions $f(t)$ and $g(t,\rho)$ respectively, then

$$\frac{1}{u}F[q(u)]G(u) = E\left\{\int_0^\infty g(t,\, \rho) f(\rho)\; d\rho\right\} \tag{10}$$

Where ρ is a parameter.

Proof: To establish the theorem, from $f(t)$ and $g(t,\rho)$ forming the integral (9) and apply Elzaki transform on (9) it reduces as

$$E\{h(t)\} = H(u) = u\int_0^\infty e^{\frac{-\rho}{u}} h(t)\, dt = u\int_0^\infty e^{\frac{-\rho}{u}} \left\{\int_0^\infty g(t,\, \rho) f(\rho)\; d\rho\right\} dt$$

By changing the order of integrals

$$H(u) = \int_0^\infty f(\rho) \left\{u\int_0^\infty e^{\frac{-\rho}{u}} g(t,\, \rho)\, dt\right\} d\rho = \int_0^\infty f(\rho)\, E\{g(t,\, \rho)\} d\rho$$

$$= \int_0^\infty f(\rho)\, G(u) e^{\frac{-\rho}{q(u)}} d\rho = G(u)\int_0^\infty f(\rho)\, e^{\frac{-\rho}{q(u)}} d\rho$$

$$= \frac{1}{u}F[q(u)]G(u)$$

Corollary 2.2: On taking $q(u) = u$ in (10)

$$G(u)e^{-\frac{\rho}{q(u)}} = G(u)e^{-\frac{\rho}{u}} = E\{U(t-\rho)g(t-\rho)\}$$

which further leads to-

$$\frac{1}{u}F(u)G(u) = E\{(f*g)(t)\}$$

Which is the result of convolution theorem for the Elzaki Transform [9,11].

Theorem 2.3: Let $M\{f(t)\} = F(u)$ and $M\{g(t,\rho)\} = G(u)\rho^{q(u)-1}$ are the Mellin Transforms of the function $f(t)$ and $g(t,\rho)$ respectively, then

$$F[q(u)]G(u) = M\left\{\int_0^\infty g(t,\, \rho)\, f(\rho)\, d\rho\right\} \tag{11}$$

Where, ρ is a parameter.
Proof: Using the integral (9) and apply Mellin transform, it reduces as

$$M\{h(t)\} = H(u) = \int_0^\infty t^{u-1}\, h(t)dt = \int_0^\infty t^{u-1} \left\{\int_0^\infty g(t,\, \rho)\, f(\rho)\, d\rho\right\} dt$$

By changing the order of integrals

$$M\{h(t)\} = \int_0^\infty f(\rho) \left\{\int_0^\infty t^{u-1}\, g(t,\, \rho)dt\right\} d\rho = \int_0^\infty f(\rho)\, M\{g(t,\, \rho)\}d\rho$$

$$= \int_0^\infty f(\rho)\, G(u)\rho^{q(u)-1}d\rho = G(u) \int_0^\infty f(\rho)\, \rho^{q(u)-1}d\rho$$

$$H(u) = G(u)F[q(u)]$$

$$M\{h(t)\} = H(u) = F[q(u)]G(u) = M\left\{\int_0^\infty g(t,\, \rho)\, f(\rho)\, d\rho\right\}$$

Corollary 2.3: On taking $q(u) = u$ in (11)

$$G(u)\rho^{q(u)-1} = G(u)\rho^{u-1} = M\{U(t-\rho)g(t-\rho)\}$$

which further leads to-

$$F(u)G(u) = M\{(f * g)(t)\}$$

which is the result of convolution theorem for the Mellin Transform [10,11].

Theorem 2.4: Let $F(s,u)$ and $G(s,u)e^{\frac{-q(s)\rho}{u}}$ are Natural transforms of $f(t)$ and $g(t,\, \rho)$ respectively, then

$$uF[q(s),u]G(s,u) = N^+\left\{\int_0^\infty g(t,\, \rho)\, f(\rho)\, d\rho\right\} \tag{12}$$

Where ρ is a parameter.
Proof: Using the integral (9) and taking Natural transform, it reduces as

$$N^+\{h(t)\} = H(s,u) = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}}\, h(t)dt = \frac{1}{u}\int_0^\infty e^{\frac{-st}{u}} \left\{\int_0^\infty g(t,\, \rho)\, f(\rho)\, d\rho\right\} dt$$

By changing the order of integrals

$$H(s,u) = \int_0^\infty f(\rho) \left\{\frac{1}{u}\int_0^\infty e^{\frac{-st}{u}}\, g(t,\, \rho)dt\right\} d\rho = \int_0^\infty f(\rho)\, N^+\{g(t,\, \rho)\}d\rho$$

$$= \int_0^\infty f(\rho)\, G(s,u)e^{\frac{-q(s)\rho}{u}}d\rho = G(s,u) \int_0^\infty f(\rho)\, e^{\frac{-q(s)\rho}{u}}d\rho$$

$$= uF[q(s),u]G(s,u)$$

Corollary 2.4: On taking $q(s) = s$ in (12)

$$G(s,u)e^{\frac{-q(s)\rho}{u}} = G(s,u)e^{\frac{-s\rho}{u}} = N^+\{U(t-\rho)g(t-\rho)\}$$

which further leads to-

$$uF(s,u)G(s,u) = M\{(f * g)(t)\}$$

which is the result of convolution theorem for the Natural Transform [12].

CONCLUDING REMARKS

The role of convolution theorem related to a transform is very important in finding the solutions of the related problems of computational mathematics. The study discovers the generalization of convolution theorem for Sumudu Transform, Elzaki Transform, Mellin Transform and Natural Transform, which leads the way to generate new properties of these transforms that can be beneficial in solving different problems where the relevant transform can be used. This study will help the researcher to uncover the critical areas of problem solving and applications of transforms of wider domain, that has been done for only Laplace Transform. Therefore, due to the general nature of the results obtained here we can derive number of new and known results by specifying the values of the general functions in the obtained results.

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