A Novel Approach of Elzaki Decomposition Method to Fractional Riccati Equations

Hemraj Choudhary, Mridula Purohit, Deepika Jain, Nawal Kishor Jangid

Department of Mathematics, Vivekananda Global University, Jaipur, Rajasthan, India, choudharyhemraj131095@gmail.com
Department of Mathematics, Vivekananda Global University, Jaipur, Rajasthan, India, mridula_purohit@vgu.ac.in
Department of Mathematics, Swami Keshvanand Institute of Technology, Management &Gramothan, Jaipur, Rajasthan, India, deepika.jain@skit.ac.in
Department of Mathematics, Swami Keshvanand Institute of Technology, Management &Gramothan, Jaipur, Rajasthan, India, nawal.jangid@skit.ac.in

Abstract

In this paper we will discuss the Elzaki decomposition method*(EDM)* for solving fractional Riccatidifferential equations.The Elzaki transform is used to construct the analytical solution of the fractional order equation through maple software. The Elzaki transform techniques provides a remarkable concern in science and engineering [4, 5, 6, 16, 20]. An admirable comparative study has also been made with the previous published results through Graphs.

2020 Mathematical Sciences Classification: 65H20, 35R11

Keywords: Fractional Riccati differential equation, Elzaki transform.

DOI: 10.48047/ecb/2023.12.9.204

1 Introduction

In recent years, it has turned out that many phenomena in biology, chemistry, acoustics, control theory, psychology and other areas of science can be fruitfully modeled by the use of fractional-order derivatives. That is because of the fact that a reasonable modeling of a physical phenomenon having dependence not only on the time instant but also on the prior time history can be successfully achieved by using fractional calculus [13]. Fractional differential equations (*FDEs*) have been used as a kind of model to describe several physical phenomena [1, 2, 3, 8, 14] such as damping laws, rheology, diffusion processes, and so on.

Moreover, some researchers have shown the advantageous use of the fractional calculus in themodeling and control of many dynamical systems. Besides modeling, finding accurate and proficient methods for solving *FDEs* has been an active research undertaking. Exact solutions for the majority of *FDEs* cannot be found easily, thus analytical and numerical methods must be used. Some

numerical methods for solving *FDEs* have been presented and they have their own advantages and limitations.
Many physical problems are governed by fractional differential equations (*FDEs*), and finding the solution of these equations have been the subject of many investigations in recent years. Recently, there have been a number of schemes devoted to the solution of fractional differential equations. These schemes can be broadly classified into two classes, numerical and analytical.

The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754). The book of Reid [17] includes the main theories of Riccatiequation, with implementations to random processes, optimal control, and diffusion problems [9].
Fractional Riccati differential equations arise in many fields, although discussions on the numerical methods for these equations are rare. Odibat and Momani [12] investigated a modified homotopy perturbation method for fractional Riccati differential equations. Khader [10] researched the fractional Chebyshev finite difference method for fractional Riccati differential equations. Li et al. [11] have solved this problem by quasi-linearization technique.

2 Preliminaries and Notations

Definition 2.1 *The fractional-order Riemann-Liouville derivative of the function* $y(t)$ *for* t *and order* $\alpha > 0$ *is defined as* [15]

$$D_t^{\alpha} y(t) = \frac{1}{\Gamma(\eta-\alpha)} \frac{d^{\eta}}{dt^{\eta}} \int_0^t (t)^{\eta-\alpha-1} y(u)\,du, \qquad \eta - 1 < \alpha \le \eta. \qquad (2.1)$$

Definition 2.2 *The fractional-order Caputo derivative of the function* $y(t)$ *for* t *and order* $\alpha > 0$ *is defined as* [15]

$$D_t^{\alpha} y(t) = \frac{1}{\Gamma(\eta-\alpha)} \int_0^t (t)^{\eta-\alpha-1} y^{\eta}(u)\,du, \qquad \eta - 1 < \alpha \le \eta. \qquad (2.2)$$

Definition 2.3 *The Elzaki transform of* $D_t^{\alpha} y(t)$ *is defined as* [18]

$$E\left[D_t^{\alpha} y(t)\right] = s^{-\alpha} E[y(t)] - \sum_{\theta=0}^{\eta-1} s^{\theta-\alpha-2}\, y^{(\theta)}(0), \quad \eta - 1 < \alpha \le \eta \quad (2.3)$$

3 Elzaki Decomposition Method (EDM)

Let us consider the following fractional non-linear differential equation to illustrate the *EDM*,

$$D_t^{\alpha} y(t) + N y(t) + L y(t) = \hbar(t), \ m - 1 < \alpha \le m, \ m \in \mathbb{N}, \qquad (3.1)$$

where

$$y(t) = y(0) \text{ at } t = 0 \qquad (3.2)$$

where $D_{ţ}^{\alpha}(.)$ is the fractional derivative of Caputo type, Ŋ is a linear operator and Ł shows a non-linear operator, with the derivatives of fractional order less than ɑ.
Taking the Elzaki transform of (3.1)

$$E\left(D_{ţ}^{\alpha}y(ţ)+Ŋy(ţ)+Ły(ţ)\right)=E(ħ(ţ))$$

and by using (2.3)

$$E[y(ţ)]=\vartheta^{\alpha}\sum_{k=0}^{ɱ-1}\vartheta^{2-\alpha-k}X^{k}(0)-\vartheta^{\alpha}E[Ŋy(ţ)+Ły(ţ)-ħ(ţ)] \qquad (3.3)$$

Taking inverse Elzaki transform

$$y(ţ)=H(ţ)-E^{-1}\{\vartheta^{\alpha}E[Ŋy(ţ)+Ły(ţ)]\} \qquad (3.4)$$

where $H(ţ)$ is the term arising from the source term and the prescribed initial condition. The representation of the solution (3.4) as an infinite series is given below

$$y(ţ)=\sum_{ɩ=0}^{\infty}y_{ɩ}(ţ)=y_{0}(ţ)+y_{1}(ţ)+y_{2}(ţ)+\cdots+y_{ɩ}(ţ)+\cdots \qquad (3.5)$$

let us decompose $Ly(ţ)$ in Adomian polynomials as

$$Ł(ţ)=\sum_{ɩ=0}^{\infty}A_{ɩ} \qquad (3.6)$$

where

$$A_{ɩ}=\frac{1}{ɩ!}\left[\frac{d^{ɩ}}{d\varepsilon^{ɩ}}\left\{Ł\sum_{ɩ=0}^{\infty}\left(\varepsilon^{ɩ}y_{ɩ}\right)\right\}\right]_{\varepsilon=0},\quad ɩ=0,1,2,\ldots, \qquad (3.7)$$

$$\sum_{ɩ=0}^{\infty}y_{ɩ+1}(ţ)=H(\tau)-E^{-1}\{\vartheta^{\alpha}E[Ŋ\sum_{ɩ=0}^{\infty}y_{i}(ţ)+\sum_{ɩ=0}^{\infty}A_{ɩ}]\},$$

where

$$y_{0}(ţ)=H(ţ)=E^{-1}\{\vartheta^{\alpha}\sum_{k=0}^{ɱ-1}\vartheta^{2-\alpha-k}y^{k}(0)+\vartheta^{\alpha}E[h(ţ)]\}, \qquad (3.8)$$

$$y_{ɩ+1}(ţ)=-E^{-1}\{\vartheta^{\alpha}E[Ŋy_{ɩ}(ţ)+A_{ɩ}]\},\quad ɩ\geq 1\ . \qquad (3.9)$$

4 Main Results

In this section we consider the following two problems to illustrate the application of *EDM* to obtain the solution to Fractional Riccati differential equations:

Problem 4.1 Consider the Fractional Riccati differential equation

$$D_{ţ}^{\alpha}y(ţ)=-y^{2}(ţ)+1,\quad 0<ţ\leq 1, \qquad (4.1)$$

with primary condition

$$y(0)=y_{0}=0. \qquad (4.2)$$

Taking Elzaki transform of (4.1) and then using (4.2), we get

$\frac{E[y(t)]}{s^{\alpha}} - \frac{E[y(0)]}{s^{\alpha-2}} = E[-y^2(t) + 1],$

i.e. $E[y(t)] = s^{\alpha}\{[s^2] - E[y^2(t)]\}.$ (4.3)

which on taking inverse Elzaki transform gives

$y(t) = E^{-1}\{s^{\alpha+2} - s^{\alpha}E[y^2(t)]\}.$ (4.4)

Now, on applying ElzakiDecomposition Method, we arrive at

$y_0(t) = y(0) = \frac{t^{\alpha}}{\Gamma[\alpha+1]},$ (4.5)

$y_1(t) = -E^{-1}s^{\alpha}(E[{y_0}^2(t)])$

$y_1(t) = -\frac{\Gamma[2\alpha+1]t^{3\alpha}}{[\Gamma[\alpha+1]]^2\Gamma[3\alpha+1]},$ (4.6)

$$y_2(t) = -E^{-1}(s^{\alpha}E[2y_0y_1])$$

$y_2(t) = \frac{2\Gamma[2\alpha+1]\Gamma[4\alpha+1]t^{5\alpha}}{[\Gamma[\alpha+1]]^3\Gamma[3\alpha+1]\Gamma[5\alpha+1]}$ (4.7)

Thus, the solution can be expressed in the series form as

$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots + y_i(t) + \cdots,$

$= \frac{t^{\alpha}}{\Gamma[\alpha+1]} - \frac{\Gamma[2\alpha+1]t^{3\alpha}}{[\Gamma[\alpha+1]]^2\Gamma[3\alpha+1]} + \frac{2\Gamma[2\alpha+1]\Gamma[4\alpha+1]t^{5\alpha}}{[\Gamma[\alpha+1]]^3\Gamma[3\alpha+1]\Gamma[5\alpha+1]} - \ldots,$ (4.8)

which on taking $\alpha = 1$, gives

$y(t) = \lim_{i\to\infty} y_i(t) = t - \frac{t^3}{3} + \frac{2}{15}t^5 + \ldots$. (4.9)

Problem 4.2: Consider the Fractional Riccati differential equation

$D_t^{\alpha}y(t) = 2y(t) - y^2(t) + 1, \quad 0 < t \le 1,$ (4.10)

with primary condition

$y(0) = y_0 = 0$. (4.11)

Taking Elzaki transform of (4.10) and then using (4.11), we get

$\frac{E[y(t)]}{s^{\alpha}} - \frac{E[y(0)]}{s^{\alpha-2}} = E[2y(t) - y^2(t) + 1],$ (4.12)

Which on taking inverse Elzaki transform gives

$$y(t) = E^{-1}\{s^{\alpha+2} + s^{\alpha}E[2y(t) - y^2(t)]\}\,. \tag{4.13}$$

Now, on applying Elzaki Decomposition Method, we arrive at

$$y_0(t) = y(0) = \frac{t^{\alpha}}{\Gamma[\alpha+1]} \tag{4.14}$$

$$y_1(t) = E^{-1}(s^{\alpha}E[2y_0(t) - {y_0}^2(t)])$$

$$y_1(t) = 2\frac{t^{2\alpha}}{\Gamma[2\alpha+1]} - \frac{\Gamma[2\alpha+1]t^{3\alpha}}{[\Gamma[\alpha+1]]^2\Gamma[3\alpha+1]}\ , \tag{4.15}$$

$$y_2(t) = -E^{-1}(s^{\alpha}E[2y_1 - 2y_0y_1])\ .$$

$$y_2(t) = 4\frac{t^{3\alpha}}{\Gamma[3\alpha+1]} - \frac{2\Gamma[2\alpha+1]t^{4\alpha}}{[\Gamma[\alpha+1]]^2\Gamma[4\alpha+1]} - \frac{4\Gamma[3\alpha+1]t^{4\alpha}}{\Gamma[\alpha+1]\Gamma[2\alpha+1]\Gamma[4\alpha+1]} + \frac{2\Gamma[2\alpha+1]\Gamma[4\alpha+1]t^{5\alpha}}{[\Gamma[\alpha+1]]^3\Gamma[3\alpha+1]\Gamma[5\alpha+1]}. \tag{4.16}$$

Thus, the solution can be expressed in the series form as

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots + y_i(t) + \cdots$$

$$= \frac{t^{\alpha}}{\Gamma[\alpha+1]} + 2\frac{t^{2\alpha}}{\Gamma[2\alpha+1]} - \frac{\Gamma[2\alpha+1]t^{3\alpha}}{[\Gamma[\alpha+1]]^2\Gamma[3\alpha+1]} + 4\frac{t^{3\alpha}}{\Gamma[3\alpha+1]} - \frac{2\Gamma[2\alpha+1]t^{4\alpha}}{[\Gamma[\alpha+1]]^2\Gamma[4\alpha+1]} - \frac{4\Gamma[3\alpha+1]t^{4\alpha}}{\Gamma[\alpha+1]\Gamma[2\alpha+1]\Gamma[4\alpha+1]} + \frac{2\Gamma[2\alpha+1]\Gamma[4\alpha+1]t^{5\alpha}}{[\Gamma[\alpha+1]]^3\Gamma[3\alpha+1]\Gamma[5\alpha+1]} + \ldots\ , \tag{4.17}$$

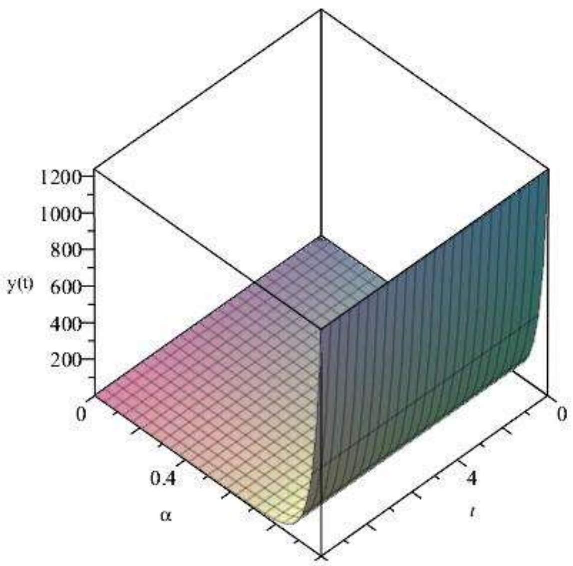
which on taking $\alpha = 1$, gives

$$y(t) = \lim_{i\to\infty} y_i(t) = t + t^2 + \frac{t^3}{3} - \frac{2t^4}{3} + \frac{2}{15}t^5 + \ldots\ . \tag{4.18}$$

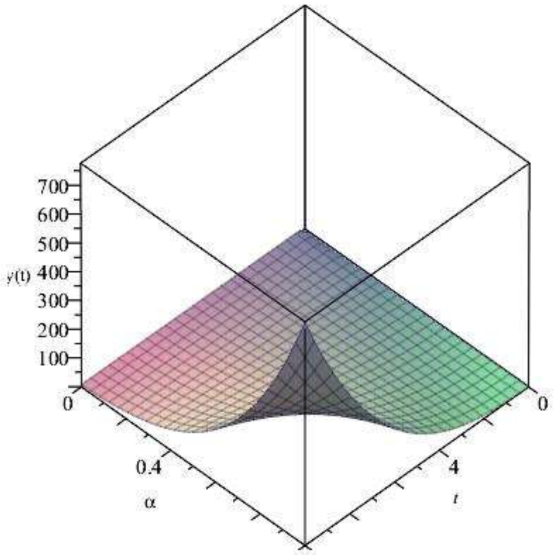
5 Graphical representation

The graphical representation of the solutions (4.8) and (4.17) with different values of γ and ε are as under:

Graph 5.1



Graph 5.2



6 Conclusion

In this article, Elzaki decomposition method has been applied for solving the fractional Riccati differential equations. An excellent study between the previously distributed outcomes and current study have also been discussed through graphs. Efficiency of the present study also determined through 3D-graphs.

References

[1] V. Feliu-Batlle, R. Perez and L. Rodriguez, Fractional robust control of main irrigation canals with variable dynamic parameters, *Control Eng. Pract.*, **15**(2007), 673-686.

[2] R. Garrappa, On some explicit Adams multistep methods for fractional differential equations. *J. Comput. Appl. Math*., **229**(2009), 392-399.

[3] M. Jamil and NA. Khan, Slip effects on fractional viscoelastic fluids, *Int. J. Differ. Equ.* 2011, Article ID **193813** (2011).

[4] S.A. Khuri, A Laplace decomposition algorithm applied to a class of nonlinear differential equations, *J. Math. Annl. Appl.*, **4**(2001), 141-155.

[5] S.A. Khuri, A new approach to Bratus problem, *Appl. Math. Comput.*, **147** (2004), 31-36.

[6] D. Kumar, J. Singh, and S. Rathore, Sumudu Decomposition Method for Nonlinear Equations, *Int. Math. For.*, **7(11)** (2012), 515 - 521.

[7] M. Khalid, M. Sultana, F. Zaidi, and U. Arshad, An Elzaki Transform Decomposition Algorithm Applied to a Class of Non-Linear Differential Equations, *J. Nat. Sci. Res.*, **5(5)**(2015), 48-55.
[8] NA. Khan, M. Jamil, A. Ara and S. Das, Explicit solution of time-fractional batch reactor system, *Int. J. Chem. React. Eng.* **9**, Article ID A91 (2011).
[9] MM. Khader, Numerical treatment for solving fractional Riccati differential equation, *J. Egypt. Math. Soc.* **21**(2013), 32-37.
[10] MM. Khader, Numerical treatment for solving fractional Riccati differential equation. *J. Egypt. Math. Soc.* **21(1)** (2013), 32-37.
[11] XY. Li, BY. Wu and RT. Wang, Reproducing kernel method for fractional Riccati differential equations, *Abstr. Appl. Anal.* **2014**, Article ID **970967** (2014)
[12] Z. Odibat, Momani, S: Modified homotopy perturbation method: application to quadratic Riccati differential
equation of fractional order. Chaos Solitons Fractals 36(1), 167-174 (2008)
[13]. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1999)
[14]. Podlubny, I: Fractional-order systems and controllers. IEEE Trans. Autom. Control 44(1), 208-214 (1999)
[15] Podlubny, I. (1999). Fractional Differential Equations. 198 Academic Press. *San Diego, California, USA.*

[16] M.S. Rawashdeh, S. Maitama, Solving Coupled System of Nonliear PDEs Using the Naturel Decomposition Method, Int. J. of Pure and Appl. Math., 92(5), (2014), 757-776.
[17]. Reid, WT: Riccati Differential Equations. Math.Sci. Eng., vol. 86. Academic Press, New York (1972)
[18] Singh, P. & Sharma, D. (2020). Comparative study of homotopy perturbation transformation with homotopy perturbation Elzaki transform method for solving nonlinear fractional PDE.

[19] Schiff, J. L. (2013). *The Laplace transform: theory and applications*. Springer Science & Business Media.

[20] D. Ziane, M. HamdiCherif, Resolution of Nonlinear Partial Differential Equations by Elzaki Transform Decomposition Method, J. Appro. Theo. Appl. Math., 5, (2015), 17-30