Program : B.Tech. in ECE

Course Name : Microwave Theory and Techniques

Course Code : 5EC4-05

Semester : V

Session : 2021-2022

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Course Plan

Total no. of lectures required =43

Unit No.	Name	No. of Lectures Required
1	Introduction: Objective, scope and outcome of the course.	1
2	Introduction to Microwaves	2
3	Mathematical Model of Microwave Transmission	3
4	Analysis of RF and Microwave Transmission	5
5	Microwave Network Analysis	4
6	Passive and Active Microwave Devices	10
7	Microwave Design Principles	8
8	Microwave Measurements	6
9	Microwave Systems	4

Syllabus

SN	Contents
1	Introduction: Objective, scope and outcome of the course.
2	Introduction to Microwaves-History of Microwaves, Microwave Frequency bands; Applications of Microwaves: Civil and Military, Medical, EMI/EMC.
3	Mathematical Model of Microwave Transmission-Concept of Mode, Features of TEM, TE and TM Modes, Losses associated with microwave transmission, Concept of Impedance in Microwave transmission.
4	Analysis of RF and Microwave Transmission Lines-Coaxial line, Rectangular waveguide, Circular waveguide, Strip line, Micro strip line.

5	Microwave Network Analysis-Equivalent voltages and currents for non TEM lines, Network parameters for microwave circuits, Scattering Parameters.
6	Passive and Active Microwave Devices-Microwave passive components: Directional Coupler, Power Divider, Magic Tee, Attenuator, Resonator. Microwave active components: Diodes, Transistors, Oscillators, Mixers. Microwave Semiconductor Devices: Gunn Diodes, IMPATT diodes, Schottky Barrier diodes, PIN diodes. Microwave Tubes: Klystron, TWT, Magnetron.
7	Microwave Design Principles-Impedance transformation, Impedance Matching, Microwave Filter Design, RF and Microwave Amplifier Design, Microwave Power Amplifier Design, Low Noise Amplifier Design, Microwave Mixer Design, Microwave Oscillator Design. Microwave Antennas- Antenna parameters, Antenna for ground based systems, Antennas for airborne and satellite borne systems, Planar Antennas.

8	Microwave Measurements-Power, Frequency and impedance measurement at microwave frequency, Network Analyzer and measurement of scattering parameters, Spectrum Analyzer and measurement of spectrum of a microwave signal, Noise at microwave frequency and measurement of noise figure. Measurement of Microwave antenna parameters.
9	Microwave Systems-Radar, Terrestrial and Satellite Communication, Radio Aidsto Navigation, RFID, GPS. Modern Trends in Microwaves Engineering- Effect of Microwaves on human body, Medical and Civil applications of microwaves, Electromagnetic interference and Electromagnetic Compatibility (EMI & EMC), Monolithic Microwave ICs, RFMEMS for microwave components, Microwave Imaging.

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Module No. 4

Analysis of RF and Microwave Transmission Lines

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Outline of the Module

- Microwave Transmission Lines
- Coaxial Transmission Lines (In this lecture)
- Rectangular Waveguide
- **Circular Waveguide (In this lecture)**
- Stripline
- Microstripline

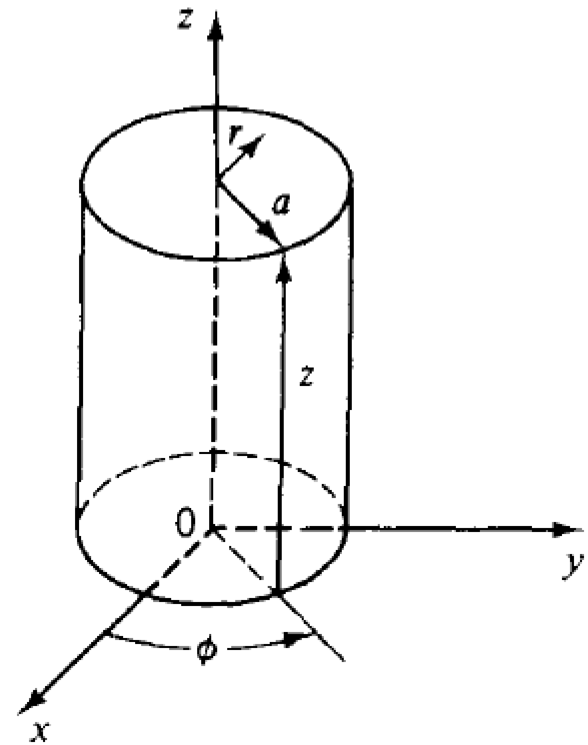
Circular Waveguide

- A circular waveguide is a tubular, circular conductor.
- A plane wave propagating through a circular waveguide results in a transverse electric (TE) or transverse magnetic (TM) mode.
- In general terms the behavior is the same as in Rectangular waveguide.

- However different geometry means different application hence a separate analysis
- The law governing the propagation of waves in waveguides are independent of the cross sectional shape and dimensions of the guide.
- All the parameters and definitions evolved for Rectangular waveguide apply to circular with minor modification

Solution of Wave equation in Cylindrical Coordinates

A cylindrical coordinate system is as shown in figure



The scalar Helmholtz equation in cylindrical coordinates is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \boldsymbol{\Psi}}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \boldsymbol{\Psi}}{\partial \emptyset^2} + \frac{\partial^2 \boldsymbol{\Psi}}{\partial z^2} = \gamma^2 \boldsymbol{\Psi}$$

Using the method of separation of variables, the solution is assumed in the form of

$$\boldsymbol{\Psi} = R(r)\emptyset(\emptyset)Z(z)$$

where R (r) = a function of the r coordinate only

$\emptyset(\emptyset)$ = a function of the $\emptyset$ coordinate only

Z (z) = a function of the z coordinate only

Substitution and division of above gives

$$\frac{1}{rR}\frac{d}{dr}(r\frac{dR}{dr}) + \frac{1}{r^2\emptyset}\frac{d^2\emptyset}{d\emptyset^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \gamma^2$$

Since the sum of the three independent terms is a constant, each of the three terms must be a constant. The third term may be set equal to a constant

$$\frac{d^2Z}{\partial z^2} = {\gamma_g}^2 z$$

The solution of this equation is given in the form

$$Z = Ae^{-\gamma_g z} + Be^{\gamma_g z}$$

$$\frac{r}{R}\frac{d}{dr}(r\frac{dR}{dr}) + \frac{1}{\emptyset}\frac{d^2\emptyset}{d\emptyset^2} - (\gamma^2 - \gamma_g{}^2)r^2 = 0$$

The second term is a function of $\emptyset$ only, hence equating the second term to a constant $(-n^2)$

$$\frac{d^2\emptyset}{d\emptyset^2} = -n^2\ \emptyset$$

The solution of this equation is also a harmonic function:

$$\emptyset = A_n \mathrm{Sin}(n\emptyset) + B_n \mathrm{Cos}(n\emptyset)$$

Replacing the $\emptyset$ term by $(-\ n^2)$ and multiplying through by *R*, we have

$$r\frac{d}{dr}(r\frac{dR}{dr}) + [(k_c r)^2 - n^2]R=0$$

This is Bessel's equation of order *n* in which

$$(\gamma_g)^2 = \gamma^2 + (k_c)^2$$

This equation is called the characteristic equation of Bessel's equation. For a lossless guide, the characteristic equation reduces to

$$\beta g = \pm\sqrt{\omega^2 \mu \epsilon - k_c^2}$$

The solutions of Bessel's equation are

$$R = C_n J_n(k_c r) + D_n N_n(k_c r)$$

where $J_n(k_c r)$ is the nth-order Bessel function of the first kind, representing a standing wave of cos $(k_c r)$ for $r < a$

$N_n(k_c r)$ is the nth-order Bessel function of the second kind, representing a standing wave of sin $(k_c r)$ for $r > a$

Therefore the total solution of the Helmholtz equation in cylindrical coordinates is given by

$$\boldsymbol{\Psi} = (C_n J_n(k_c \mathrm{r}) + D_n N_n(k_c \mathrm{r}))(A_n \mathrm{Sin}(\mathrm{n}\emptyset) + B_n \mathrm{Cos}(\mathrm{n}\emptyset))e^{\pm j\beta_g z}$$

From the Bessel functions graphs, at $r = 0$, however, $k_c\mathrm{r} = 0$, then the function N. approaches infinity, so $D_n = 0$. Also, by use of trigonometric manipulations, the two sinusoidal terms become combined

Finally, the solution of the Helmholtz equation is reduced to

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}_0 (J_n(k_c \mathrm{r}))\mathrm{Cos}(\mathrm{n}\emptyset)e^{-j\beta_g z}$$

TE mode in Circular Waveguide

The *TE* modes in a circular waveguide are characterized by $E_z = 0$. In other words, the z component of the magnetic field, H_z, must exist in order to have energy transmission in the guide. Consequently, from a given Helmholtz equation, H_z is not equal to zero, so:

$$\nabla^2 H_z = \gamma^2 H_z$$

Its solution is of the form

$$\boldsymbol{H}_z = \boldsymbol{H}_{0z}(J_n(k_c \mathrm{r}))\mathrm{Cos}(\mathrm{n}\emptyset)e^{-j\beta_g z}$$

The Maxwell equations for a lossless dielectric medium becomes in frequency domain as:

$$\nabla X E = -j\omega\,\mu H$$
$$\nabla X H = j\omega\varepsilon E$$

Suppose, i,j,k are unit vectors along X, Y and Z directions

Electric field Vector E= $E_x\, i + E_y\, j + E_z\, k$

Magnetic field Vector H= $H_x\, i + H_y\, j + H_z\, k$

We expand the curl equations in cylindrical coordinates

The boundary conditions require that the $\emptyset$ component of the electric field $E_{\emptyset}$, which is tangential to the inner surface of the circular waveguide at $r = a$, must vanish or that the r component of the magnetic field H_r, which is normal to the inner surface of $r = a$, must vanish. Consequently

$E_{\emptyset}= 0$, at r=a, then $\frac{\partial H_z}{\partial r} = 0$ at r=a

$$\boldsymbol{H}_{0z}(J_n'(k_c a))\mathrm{Sin}(n\emptyset)e^{-j\beta_g z}=0$$

$$J_n'(k_c a)=0$$

The permissible values are

$$k_c = \frac{X_{np}'}{a}$$

- $E_r = \mathrm{E_{0r}}\, J_n(\frac{X'_{np}r}{a})\mathrm{Sin(n\emptyset)e^{-j\beta_g z}}$

- $E_\emptyset = E_{0\emptyset}\, J'_n\left(\frac{X'_{np}r}{a}\right)\mathrm{Cos(n\emptyset)e^{-j\beta_g z}}$

- $E_z = 0$

- $H_r = H_{0r} J_n\left(\frac{X'_{np} r}{a}\right) \mathrm{Cos}(n\emptyset) e^{-j\beta_g z}$

- $H_\emptyset = H_{0\emptyset} J_n\left(\frac{X'_{np} r}{a}\right) \mathrm{Sin}(n\emptyset) e^{-j\beta_g z}$

- $H_z = H_{0z} J_n\left(\frac{X'_{np} r}{a}\right) \mathrm{Cos}(n\emptyset) e^{-j\beta_g z}$

n=0,1,2,3 and p=1,2,3,4

The first subscript *n* represents the number of full cycles of field variation in one revolution through 2π rad of $\emptyset$. The second subscript *p* indicates the number of zeros of $E_{\emptyset}$,-that is, $J_n\left(\frac{X'_{np}r}{a}\right)$ along the radial of a guide, but the zero on the axis is excluded if it exists.

Mode propagation constant

$$\beta g = \sqrt{\omega^2 \mu \epsilon - \left(\frac{X'_{np}}{a}\right)^2}$$

The cutoff wave number of a mode is that for which the mode propagation constant vanishes. Hence

$$k_c = \frac{X'_{np}}{a} = \omega_c \sqrt{\mu \epsilon}$$

The cutoff frequency for TE modes in a circular guide is then given by

$$f_c = \frac{X'_{np}}{2\pi a \sqrt{\mu \epsilon}}$$

$$v_p = \frac{\omega}{\beta_g}$$ (Phase velocity)

- $$v_p = \frac{c}{\sqrt{1 - \left(\frac{fc}{f}\right)^2}}$$
- $$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{fc}{f}\right)^2}}$$
- $$z_g = \frac{\omega\mu}{\beta_g} = \frac{\eta}{\sqrt{1 - \left(\frac{fc}{f}\right)^2}}$$

TM modes in Circular Waveguide

The *TM* modes in a circular waveguide are characterized by $H_z = 0$. In other words, the *z* component of the electric field, E_z, must exist in order to have energy transmission in the guide. Consequently, from a given Helmholtz equation, H_z is not equal to zero, so:

$$\nabla^2 E_z = \gamma^2 E_z$$

Its solution is of the form

$$\boldsymbol{E}_z = \boldsymbol{E}_{0z}(J_n(k_c \mathrm{r}))\mathrm{Cos}(\mathrm{n}\emptyset)e^{-j\beta_g z}$$

The boundary condition requires that the tangential component of electric field E_z at $r = a$ vanishes. Consequently,

$$J_n(k_c a)=0$$

The Maxwell equations for a lossless dielectric medium becomes in frequency domain as:

$$\nabla X E = -j\omega\ \mu H$$
$$\nabla X H = j\omega\ \varepsilon\ E$$

Suppose, i,j,k are unit vectors along X, Y and Z directions

Electric field Vector E= $E_x\ i + E_y\ j + E_z\ k$

Magnetic field Vector H= $H_x\ i + H_y\ j + H_z\ k$

We expand the curl equations in cylindrical coordinates

- $E_r = \mathrm{E_{0r}}\, J_n'\left(\frac{X_{np}r}{a}\right) \mathrm{Cos(n\emptyset)e^{-j\beta_g z}}$

- $E_\emptyset = E_{0\emptyset}\, J_n\left(\frac{X_{np}r}{a}\right) \mathrm{Sin(n\emptyset)e^{-j\beta_g z}}$

- $E_z = E_{0z}\, J_n\left(\frac{X_{np}r}{a}\right) \mathrm{Cos(n\emptyset)e^{-j\beta_g z}}$

- $H_r = H_{0r} J_n \left(\frac{X_{np} r}{a}\right) \mathrm{Sin(n\emptyset)e^{-j\beta_g z}}$

- $H_\emptyset = H_{0\emptyset}\, J_n' \left(\frac{X_{np} r}{a}\right) \mathrm{Cos(n\emptyset)e^{-j\beta_g z}}$

- $H_z = 0$

Mode propagation constant

$$\beta g = \sqrt{\omega^2 \mu \epsilon - \left(\frac{X_{np}}{a}\right)^2}$$

The cutoff wave number of a mode is that for which the mode propagation constant vanishes. Hence

$$k_c = \frac{X_{np}}{a} = \omega_c \sqrt{\mu \epsilon}$$

The cutoff frequency for TE modes in a circular guide is then given by

$$f_c = \frac{X_{np}}{2\pi a \sqrt{\mu \epsilon}}$$

$$v_p = \frac{\omega}{\beta_g} \text{ (Phase velocity)}$$

- $v_p = \dfrac{c}{\sqrt{1-\left(\frac{fc}{f}\right)^2}}$
- $\lambda_g = \dfrac{\lambda}{\sqrt{1-\left(\frac{fc}{f}\right)^2}}$

$$z_g = \frac{\beta_g}{\omega\epsilon} = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

It should be noted that the dominant mode, or the mode of lowest cutoff frequency in a circular waveguide, is the mode of TE_{11} that has the smallest value of the product, k_ca = 1. 841

TEM mode in circular waveguide

- TEM modes are characterized by both E_z and H_z as zero
- This means that the electric and magnetic fields are completely transverse to the direction of wave propagation.
- This mode cannot exist in hollow waveguides, since it requires two conductors, such as the coaxial transmission line and two-open-wire line.

Summary of the lecture

- In this lecture, we covered mathematical analysis of Circular Waveguides, it is somewhat similar to that of rectangular waveguides, with calculations in cylindrical coordinate sytem make mathematics more complex
- In the next lecture, we will start with analysis of Microstriplines and Striplines

References

- NPTEL course "Basic Building Blocks of Microwave Engineering"IIT, Kharagpur
- D.M. Pozar, Microwave Engineering, John Wiley & Sons, 2012.
- S.Y. Liao, Microwave Devices and Circuits, Prentice Hall, 1987

THANK YOU !!

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Waveguide Analysis

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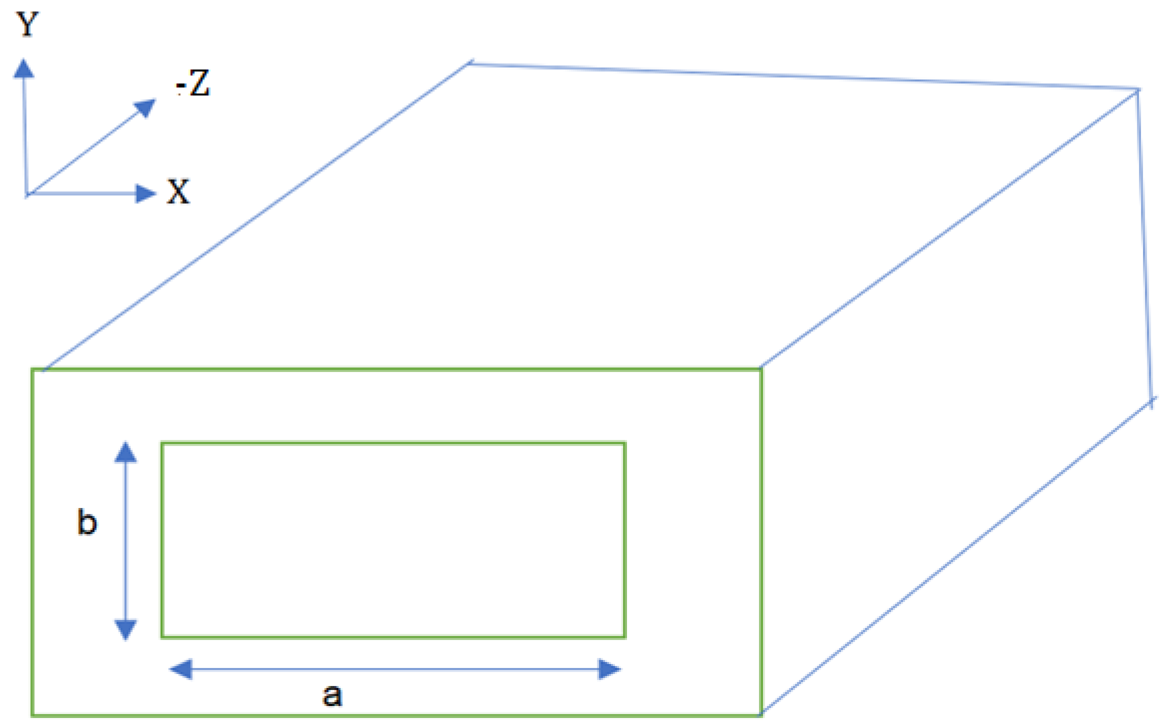
Solution of Wave Equations in Rectangular Coordinates

The process of analyzing the waveguide involves following steps:

- The desired wave equations are written in the form of either rectangular or cylindrical coordinate systems as required
- The boundary conditions are then applied to the wave equations
- The resultant equations are in the form of partial differential equations which can be solved by using the proper method

A rectangular coordinate system is as shown in figure, with a rectangular waveguide having wave propagation along -z direction



As per the Helmholtz equation:

$$\nabla^2 \Psi = \gamma^2 \Psi$$

Where the function Ψ can be written as a function of x, y and z as:

$$\Psi = X(x)Y(y)Z(z)$$

by using separation of variables

Gamma is propagation constant as discussed previously

$$\gamma = \sqrt{(\, j\omega\, \mu(\sigma + j\omega\, \epsilon))}$$

Helmholtz equation in rectangular coordinates is given by:

$$\frac{\partial^2 \boldsymbol{\Psi}}{\partial \mathrm{x}^2} + \frac{\partial^2 \boldsymbol{\Psi}}{\partial \mathrm{y}^2} + \frac{\partial^2 \boldsymbol{\Psi}}{\partial \mathrm{z}^2} = \gamma^2 \boldsymbol{\Psi}$$

Substitution of $\boldsymbol{\Psi} = \mathbf{X}(\mathbf{x})\mathbf{Y}(\mathbf{y})\mathbf{Z}(\mathbf{z})$

Gives

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = \gamma^2$$

Since the sum of the three terms on the left-hand side is a constant and each term is independently variable, it follows that each term must be equal to a constant.

Let the three terms be k_x^2, k_y^2, k_z^2, respectively, then the separation equation becomes

$$-k_x^2 - k_y^2 - k_z^2 = \gamma^2$$

where

$$-k_x^2 = \frac{1}{X}\frac{d^2X}{dx^2}$$

$$-k_y^2 = \frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$-k_z^2 = \frac{1}{Z}\frac{d^2Z}{dz^2}$$

The general solution of equation will be:

$$X = A\ \mathrm{Sin}(k_x x) + B\ \mathrm{Cos}(k_x x)$$

$$Y = C\ \mathrm{Sin}(k_y y) + D\ \mathrm{Cos}(k_y y)$$

$$Z = E\ \mathrm{Sin}(k_z z) + F\ \mathrm{Cos}(k_z z)$$

The total solution of the Helmholtz equation in rectangular coordinates is:

$$\boldsymbol{\Psi} = (A\ \mathrm{Sin}(k_x x) + B\ \mathrm{Cos}(k_x x))(C\ \mathrm{Sin}(k_y y) + D\ \mathrm{Cos}(k_y y))(E\ \mathrm{Sin}(k_z z) + F\ \mathrm{Cos}(k_z z))$$

The propagation of the wave in the guide is assumed in the *z* direction, the propagation constant γ_g in the guide differs from the intrinsic propagation constant γ of the dielectric as:

$$(\gamma_g)^2 = \gamma^2 + (k_x)^2 + (k_y)^2 = \gamma^2 + (k_c)^2$$

where

$$k_c = \sqrt{k_x^2 + k_y^2}$$

is called cut off wave number

For a lossless dielectric, $\sigma=0$, So,

$$\gamma^2 = -\omega^2 \mu\epsilon$$

Therefore,

$$\gamma_g = \pm\sqrt{-\omega^2 \mu\epsilon + (k_c)^2}$$

There are three cases for the propagation constant γ_g in the waveguide

Case I (Cut off case)

- $\omega^2 \mu\epsilon = k_c^2$
- $\gamma_g = 0$

No propagation, the cut off frequency in this case is given as:

- $\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{k_x^2 + k_y^2}$
- $f_c = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{k_x^2 + k_y^2}$

Case II (Propagation Case)

- $\omega^2 \mu\epsilon > k_c^2$
- $\gamma_g = \pm j\,\omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{fc}{f}\right)^2}$
- $\gamma_g = \pm j\beta g = \pm j\,\omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{fc}{f}\right)^2}$

Here, the attenuation is zero and wave will propagate if:

- f > fc

Case III (Attenuation Case)

- $\omega^2 \mu \epsilon < k_c^2$
- $\gamma_g = \pm\, \omega\sqrt{\mu\epsilon}\sqrt{\left(\frac{fc}{f}\right)^2 - 1}$
- $\gamma_g = \pm\, \alpha g = \pm\omega\sqrt{\mu\epsilon}\sqrt{\left(\frac{fc}{f}\right)^2 - 1}$

Here, the attenuation is non zero but phase constant is zero so wave will not propagate if:

- $f < fc$

So, considering the previous equation

$$-k_z^2 = \frac{1}{Z}\frac{d^2Z}{dz^2}$$

Its solution can also be written in the form as:

$$Z = e^{-jk_z z}$$

as the wave is propagating in -z direction also, k_z is replaced by β_g

and the solution of Ψ becomes

$$\Psi = (A\ \text{Sin}(k_x x) + B\ \text{Cos}(k_x x))(C\ \text{Sin}(k_y y) + D\ \text{Cos}(k_y y))\ e^{-j\beta_g z}$$

Representation of Modes

The variables k_x and k_y can be written as :

$$k_x = \frac{m\pi}{a}$$

$$k_y = \frac{n\pi}{b}$$

The general symbol of representation will be TE m, n or TM m, n where: where m,n are integers as 0,1,2……

the subscript 'm' indicates the number of half wave variations of the electric field intensity along the a (wide) dimension of the waveguide., the second subscript 'n' indicates the number of half wave variations of the electric field in the b (narrow) dimension of the guide.

The TE10 mode has the longest operating wavelength and is designated as the **dominant mode.** It is the mode for which the lowest frequency that can be propagated in a waveguide.

Phase Velocity and Group Velocity

$$\beta g = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{fc}{f}\right)^2}$$

$$v_p = \frac{\omega}{\beta_g}$$

$$v_g = \frac{d\omega}{d\beta_g}$$

$$v_p = \frac{c}{\sqrt{1 - \left(\frac{fc}{f}\right)^2}}$$

$$v_g = c\sqrt{1 - \left(\frac{fc}{f}\right)^2}$$

Relation between Wavelengths

$$\beta g = \omega\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{fc}{f}\right)^2}$$

$$\beta g = \frac{2\pi}{\lambda_g} = 2\pi f\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{fc}{f}\right)^2}$$

$$\frac{1}{(\lambda)^2} = \frac{1}{(\lambda g)^2} + \frac{1}{(\lambda c)^2}$$

λ is the free space wavelength

λg is the guided wavelength

λc is the cut off wavelength

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TE and TM modes in rectangular waveguides

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Rectangular Waveguides

TE Modes:

The *TE* modes in a rectangular waveguide are characterized by $E_z = 0$. In other words, the *z* component of the magnetic field, H_z, must exist in order to have energy transmission in the guide. Consequently, from a given Helmholtz equation, H_z is not equal to zero, so:

$$\nabla^2 H_z = \gamma^2 H_z$$

A solution will be of the form

$$H_z = (\text{Am Sin}\left(\frac{m\pi x}{a}\right) + \text{Bm Cos}\left(\frac{m\pi x}{a}\right))(\text{Cn Sin}\left(\frac{n\pi y}{b}\right) + \text{Dn Cos}\left(\frac{n\pi y}{b}\right))\, e^{-j\beta_g z}$$

The Maxwell equations for a lossless dielectric medium becomes in frequency domain as:
∇XE = −jω μH
∇XH = jωЄE
Suppose, i,j,k are unit vectors along X, Y and Z directions
Electric field Vector E= $E_x\,i + E_y\,j + E_z\,k$
Magnetic field Vector H= $H_x\,i + H_y\,j + H_z\,k$
On expanding the curl equations

The Maxwell equations for a lossless dielectric medium becomes in frequency domain as:

$$\nabla X E = -j\omega\ \mu H$$
$$\nabla X H = j\omega \epsilon E$$

Suppose, i,j,k are unit vectors along X, Y and Z directions

Electric field Vector E= $E_x\,i + E_y\,j + E_z\,k$

Magnetic field Vector H= $H_x\,i + H_y\,j + H_z\,k$

On expanding the curl equations

- $$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\,\mu H_x$$
- $$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\,\mu H_y$$
- $$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\,\mu H_z$$
- $$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\,\epsilon E_x$$
- $$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\,\epsilon E_y$$
- $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\,\epsilon E_z$$

If we assume exponential variation of fields with z then,

$\frac{\partial}{\partial z}$ can be replaced by $-j\beta_g$

E_z=0 for TE modes

Also,

from, Case II (Propagation Case)

- $\omega^2 \mu \epsilon > k_c^2$
- $\gamma_g = \pm j\beta g = \sqrt{k_c^2 - \omega^2 \mu \epsilon}$
- $k_c^2 = \omega^2 \mu \epsilon - \beta_g^2$

- $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\,\mu H_x$
- $\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\,\mu H_y$
- $\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\,\mu H_z$
- $\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\,\epsilon E_x$
- $\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\,\epsilon E_y$
- $\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\,\epsilon E_z$

- $E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$
- $E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$
- $E_z = 0$
- $H_x = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial x}$
- $H_y = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial y}$

$$H_z = (\text{Am Sin}(\frac{m\pi x}{a}) + \text{Bm Cos}(\frac{m\pi x}{a}))(\text{Cn Sin}(\frac{n\pi y}{b}) + \text{Dn Cos}(\frac{n\pi y}{b}))\, e^{-j\beta_g z}$$

The boundary conditions are applied to the field equations such that the tangent E field is zero at a surface

- Since $E_x = 0$, then $\frac{\partial H_z}{\partial y} = 0$ at $y = 0$, b. Hence Cn = 0.
- Since $E_y = 0$, then $\frac{\partial H_z}{\partial x} = 0$ at $x = 0, a$ Hence Am=0

Therefore the magnetic field is given by:

$$H_z = H_{0z} \mathrm{Cos}\left(\frac{m\pi x}{a}\right) \mathrm{Cos}\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z}$$

H_{0z} is a constant

On substitution of H_z the other components become

- $E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$
- $E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$
- $E_z = 0$

- $H_x = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial x}$
- $H_y = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial y}$

- $E_x = E_{0x} Cos(\frac{m\pi x}{a}) Sin(\frac{n\pi y}{b}) e^{-j\beta_g z}$

- $E_y = E_{0y} Sin(\frac{m\pi x}{a}) Cos(\frac{n\pi y}{b}) e^{-j\beta_g z}$

- $E_z = 0$

- $H_x = H_{0x} Sin(\frac{m\pi x}{a}) Cos(\frac{n\pi y}{b}) e^{-j\beta_g z}$

- $H_y = H_{0y} Cos(\frac{m\pi x}{a}) Sin(\frac{n\pi y}{b}) e^{-j\beta_g z}$

- $H_z = H_{0z} Cos(\frac{m\pi x}{a})\ Cos(\frac{n\pi y}{b}) e^{-j\beta_g z}$

The cutoff wave number k_c. as defined for the TE_{mn} modes, is given by

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \omega_c\sqrt{\mu\epsilon}$$

where *a* and *b* are in meters. The cut off frequency, for the TE_{mn} modes, is

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

The propagation constant (or the phase constant here) is expressed by

$$\beta_g = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

The characteristic wave impedance of TE_{mn} modes in the guide can be derived

$$z_g = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta_g} = \frac{\eta}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$$

- $E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$
- $H_y = \frac{-j\beta_g}{k_c^2}\frac{\partial H_z}{\partial y}$

The wavelength in the guide for the TE_{mn} modes is given by

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$\beta_g = \frac{2\pi}{\lambda_g} = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

TM Modes:

The TM modes in a rectangular waveguide are characterized by $H_z = 0$. In other words, the z component of the magnetic field, E_z, must exist in order to have energy transmission in the guide. Consequently, from a given Helmholtz equation, E_z is not equal to zero

$$\nabla^2 E_z = \gamma^2 E_z$$

A solution will be of the form

$$E_z = (\text{Am Sin}(\frac{m\pi x}{a}) + \text{Bm Cos}(\frac{m\pi x}{a}))(\text{Cn Sin}(\frac{n\pi y}{b}) + \text{Dn Cos}(\frac{n\pi y}{b}))\, e^{-j\beta_g z}$$

The boundary conditions are applied to the field equations such that the tangent E field is zero at a surface

$E_z = 0$ at $x = 0, a$ then $Bm = 0$,

and for $E_z = 0$ at $y = 0, b$ then $Dn = 0$

$$E_z = (\text{Am Sin}(\frac{m\pi x}{a}) + \text{Bm Cos}(\frac{m\pi x}{a}))(\text{Cn Sin}(\frac{n\pi y}{b}) + \text{Dn Cos}(\frac{n\pi y}{b}))\ e^{-j\beta_g z}$$

Therefore the electric field is given by:

$$E_z = E_{0z} \mathrm{Sin}\left(\frac{m\pi x}{a}\right) \mathrm{Sin}\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z}$$

E_{0z} is a constant

If either $m = 0$ or $n = 0$, the field intensities all vanish. So there is no TM_{01} or TM_{10} mode in a rectangular waveguide

On again expanding the curl of equations

$$\nabla X E = -j\omega\,\mu H$$

$$\nabla X H = j\omega \epsilon E$$

We have

- $$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\, \mu H_x$$
- $$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\, \mu H_y$$
- $$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\, \mu H_z$$
- $$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\, \epsilon E_x$$
- $$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\, \epsilon E_y$$
- $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\, \epsilon E_z$$

If we assume exponential variation of fields with z then, $\frac{\partial}{\partial z}$ can be replaced by $-j\beta_g$

$H_z = 0$ for TM modes

Also,

$$k_c^2 = \omega^2 \mu\epsilon - \beta_g^2$$

- $$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\,j\omega\,\mu H_x$$

- $$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\,j\omega\,\mu H_y$$

- $$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\,j\omega\,\mu H_z$$

- $$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\,\epsilon E_x$$

- $$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\,\epsilon E_y$$

- $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\,\epsilon E_z$$

- $H_x = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y}$

- $H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial x}$

- $H_z = 0$

- $E_x = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial x}$

- $E_y = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial y}$

$$E_z = E_{0z} \mathrm{Sin}\left(\frac{m\pi x}{a}\right) \mathrm{Sin}\left(\frac{n\pi y}{b}\right) e^{-j\beta_g z}$$

On substituting E_z the other components will become

- $H_x = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y}$
- $H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial x}$
- $H_z = 0$

- $E_x = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial x}$

- $E_y = \frac{-j\beta_g}{k_c^2}\frac{\partial E_z}{\partial y}$

- $E_x = E_{0x} Cos(\frac{m\pi x}{a}) Sin(\frac{n\pi y}{b}) e^{-j\beta_g z}$
- $E_y = E_{0y} Sin(\frac{m\pi x}{a}) Cos(\frac{n\pi y}{b}) e^{-j\beta_g z}$
- $E_z = E_{0z} Sin(\frac{m\pi x}{a}) Sin(\frac{n\pi y}{b}) e^{-j\beta_g z}$
- $H_x = H_{0x} Sin(\frac{m\pi x}{a}) Cos(\frac{n\pi y}{b}) e^{-j\beta_g z}$
- $H_y = H_{0y} Cos(\frac{m\pi x}{a}) Sin(\frac{n\pi y}{b}) e^{-j\beta_g z}$
- $H_z = 0$

The cutoff wave number k_c. as defined for the TM_{mn} modes, is given by

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \omega_c\sqrt{\mu\epsilon}$$

where a and b are in meters. The cutoff frequency, for the TM_{mn} , same as that for TE modes is

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

The propagation constant (or the phase constant here) is expressed by

$$\beta_g = \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

The characteristic wave impedance of TM_{mn} modes in the guide can be derived

$$z_g = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta_g}{\omega \epsilon} = \eta \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

- $H_y = \frac{-j\omega\epsilon}{k_c^2} \frac{\partial E_z}{\partial x}$
- $E_x = \frac{-j\beta_g}{k_c^2} \frac{\partial E_z}{\partial x}$

The wavelength in the guide for the TM_{mn} modes is given by

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

TEM Mode:

Considering, curl equations as before

- $$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\, \mu H_x$$
- $$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\, \mu H_y$$
- $$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\, \mu H_z$$
- $$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\, \epsilon E_x$$
- $$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\, \epsilon E_y$$
- $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\, \epsilon E_z$$

- TEM modes are characterized by both E_z and H_z as zero
- By putting this all the field components become zero, so TEM modes do not exist in waveguides

- $$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\, \mu H_x$$
- $$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\, \mu H_y$$
- $$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\, \mu H_z$$
- $$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\, \epsilon E_x$$
- $$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\, \epsilon E_y$$
- $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\, \epsilon E_z$$

Also, theoretically TEM modes exist only in the presence of more than one conductor but waveguides have a single conductor so, TEM modes do not exist in waveguides

- Whenever two or more modes have the same cutoff frequency, they are said to be degenerate modes.
- In a rectangular waveguide the corresponding TE_{mn} and TM_{mn} modes are always degenerate
- The TE_{10} mode has the longest operating wavelength and is designated as the dominant mode. It is the mode for the lowest cut off frequency that can be propagated in a waveguide
- For TM modes the dominant mode is TM_{11}